

Weak Unit Disk and Interval Representation of Planar Graphs

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Abstract. We study a variant of intersection representations with unit balls, that is, unit disks in the plane and unit intervals on the line. Given a planar graph and a bipartition of the edges of the graph into *near* and *far* sets, the goal is to represent the vertices of the graph by unit balls so that the balls representing two adjacent vertices intersect if and only if the corresponding edge is near. We consider the problem in the plane and prove that it is NP-hard to decide whether such a representation exists for a given edge-partition. On the other hand, every series-parallel graph admits such a representation with unit disks for any near/far labeling of the edges. We also show that the representation problem on the line is equivalent to a variant of a graph coloring. We give examples of girth-4 planar and girth-3 outerplanar graphs that have no such representation with unit intervals. On the other hand, all triangle-free outerplanar graphs and all graphs with maximum average degree less than $26/11$ can always be represented. In particular, this gives a simple proof of representability of all planar graphs with large girth.

1 Introduction

Intersection graphs of various geometric objects have been extensively studied for their many applications [15]. A graph is a d -dimensional *unit ball graph* if its vertices are represented by unit size balls in \mathbb{R}^d , and an edge exists between two vertices if and only if the corresponding balls intersect. Unit ball graphs are called *unit disk graphs* when $d = 2$ and *unit interval graphs* when $d = 1$. In this paper we study the so called *weak unit ball graphs*. Given a graph G whose edges have been partitioned into “near” and “far” sets, we wish to assign unit balls to the vertices of G so that, for an edge (u, v) of G , the balls representing u and v intersect if and only if the edge (u, v) is near. Note that if (u, v) is not an edge of G , then the balls of u and v may or may not intersect. We refer to such graphs as *weak unit disk* or *weak unit interval graphs* when $d = 2$ or $d = 1$, respectively. A geometric representation of such graphs (particularly, a mapping of the vertices to \mathbb{R}^2 or \mathbb{R}), is called a *weak unit disk drawing* or a *weak unit interval drawing*; see Fig. 1. In figures near edges are shown as thick line segments and far edges are dashed line segments. Unit disk drawings allow us to represent the edges of a graph by proximity, which is intuitive from human perception point of view. Weak unit disk graphs also allow to arbitrarily forbid edges between certain pairs of vertices, which may be useful in drawing “almost” unit disk graphs. It has been shown that weak unit interval graphs can be used to compute *unit-cube contact representations* of planar graphs [4]; see Appendix A for more details.

Unit disk graphs have been extensively studied for their application to wireless sensor and radio networks. In such a network, we can model each sensor or radio as a device with a unit size broadcast range, which naturally induces a unit disk graph by adding an edge whenever two ranges intersect (or equivalently, when the range of one node contains a second node). This setting makes it easy to study various practical problems. For example, in the *frequency assignment problem*, we wish to assign frequencies to radio towers so that nearby towers do not interfere with each other. A weakness of the unit disk model is that it does not allow for interference between nodes from weather and geography, and it does not account for the possibility that a pair of nodes may not be able to communicate due to technological barriers. One attempt to address this issue has been the *quasi unit disk graph* [16], where each vertex is represented by a pair of disks, one of radius r , $0 < r < 1$, and the other of radius 1. In this model, two vertices are connected by an edge if their radius- r disks overlap, and do not have an edge if their radius-1 disks do not overlap. The remaining edges are in or out of the graph on a case by case basis. In the weak unit disk model, such problems can be dealt with by simply deleting edges between nodes which are nearby but nevertheless whose ranges do not

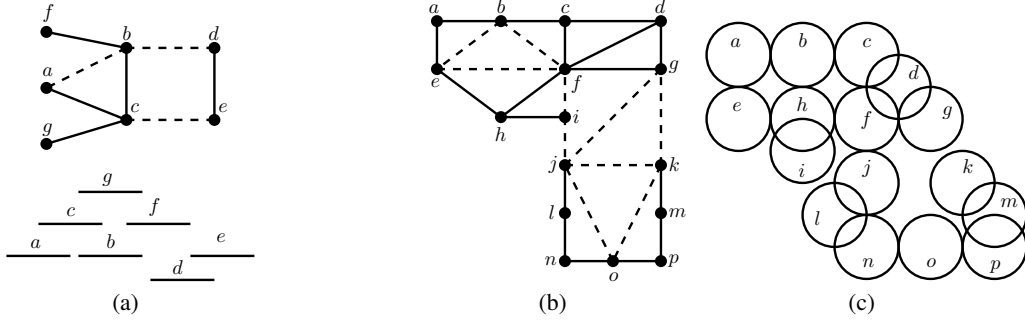


Fig. 1. (a) A planar graph with an edge-labeling and its weak unit interval representation. (b-c) A graph with its weak unit disk representation. In the figures we indicate near edges with solid lines and far edges with dashed lines.

overlap (for example because they are separated by a mountain range). This gives us more flexibility than quasi unit disk graphs.

Formally, an *edge-labeling* of a graph $G = (V, E)$ is a map $\ell : E \rightarrow \{N, F\}$. If $(u, v) \in E$, then (u, v) is called *near* if $\ell(u, v) = N$, and otherwise (u, v) is called *far*. In a unit disk (interval) representation I , each vertex $v \in V$ is represented as a disk (interval) centered at the point $I(v) \in \mathbb{R}^2$ (\mathbb{R}). We denote by $\|I(u) - I(v)\|$ the distance between the points $I(u)$ and $I(v)$, and by a slight abuse of notation, we also refer to $I(v)$ as the disk (interval) representing $v \in V$. A weak unit disk (interval) representation of G with respect to ℓ is a representation I such that for each edge $(u, v) \in E$, $\|I(u) - I(v)\| \leq d$ if and only if $\ell(u, v) = N$, for some fixed unit $d > 0$ (in other words, the disks and intervals have diameter d). Unless otherwise stated, we assume $d = 1$. In this paper, we study *weak unit disk (interval) planar graphs*, that is, planar graphs that have appropriate representations for all possible edge-labelings.

1.1 Related Work

Weak unit disk graphs can be seen as a form of graph drawing/labeling where closeness between vertices is used to define edges, albeit only from a defined set of permissible edges. There have been many classes of graphs, defined on some notion of closeness of the vertices. *Proximity graphs* are ones that can be drawn in the plane such that every pair of adjacent vertices satisfies some fixed notion of closeness, whereas every pair of non-adjacent vertices satisfy some notion of farness. A common class of proximity graphs are *Gabriel graphs*, first applied in [12] to the categorization of biological populations. Gabriel graphs can be embedded in the plane so that, for every pair of vertices (u, v) , the disk with u and v as antipodal points contains no other vertex if and only if (u, v) is an edge. Recently, Evans et al. [9] studied *region of influence graphs*, where each pair of vertices u, v in the plane are assigned a region $R(u, v)$, and there is an edge if and only if $R(u, v)$ contains no vertices except possibly u and v . They generalized this class of graphs to *approximate proximity graphs*, where there are parameters $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that a vertex other than u or v is contained in $R(u, v)$, scaled by $1/(1 + \epsilon_1)$ in an appropriate fashion, if and only if (u, v) is an edge, and the region $R(u, v)$, scaled by $1 + \epsilon_2$ in an appropriate fashion, is empty if and only if (u, v) is not an edge. Such graphs place a stronger requirement on how far away non-adjacent vertices must be than typical proximity graphs.

Weak unit ball representability in 1D is equivalent to *threshold-coloring* [1]. In this variant of the graph coloring, integer colors are assigned to the vertices so that endpoints of near edges differ by less than a given threshold, while endpoints of far edges differ by more than the threshold. It is shown that deciding whether a graph is threshold-colorable with respect to a given partition of edges into near and far is equivalent to the graph sandwich problem for proper-interval-representability, which is known to be NP-complete [13]. Hence, deciding if a graph admits a weak unit interval representation with respect to a given edge-labeling

is also NP-complete. Note the difference with recognizing unit interval graphs, which can be done in linear time [10]. It is also known that planar graphs with girth (the length of a shortest cycle in the graph) at least 10 are always threshold-colorable. Several Archimedean lattices (which correspond to tilings of the plane by regular polygons), and some of their duals, the Laves lattices, are threshold-colorable [2]. Hence, these graph classes are weak unit interval graphs.

Unit interval graphs are also related to threshold and difference graphs. In *threshold graphs* there exists a real number S and for every vertex v there is a real weight a_v so that (v, w) is an edge if and only if $a_v + a_w \geq S$ [17]. A graph is a *difference graph* if there is a real number S and for every vertex v there is a real weight a_v so that $|a_v| < S$ and (v, w) is an edge if and only if $|a_v - a_w| \geq S$ [14]. Note that for both these classes the existence of an edge is completely determined by the threshold S , while in our setting the edges defined by the threshold (size of the ball) must also belong to the original (not necessarily complete) graph. Threshold-colorability is also related to the *integer distance graph* representation [8, 11]. An integer distance graph is a graph with the set of integers as vertex set and with an edge joining two vertices u and v if and only if $|u - v| \in D$, where D is a subset of the positive integers. Clearly, an integer distance graph is threshold-colorable if the set D is a set of consecutive integers.

1.2 Our Results

We introduce the notion of weak unit disk and interval representations. While finding representations with unit intervals is equivalent to threshold-coloring and so some results are already known, the problem of weak unit disk representability is new. We first show that recognizing weak unit disk graphs is hard: For a graph G with an edge-labeling ℓ , it is NP-hard to decide if G has a weak unit disk representation with respect to ℓ , even if the edges labeled N induce a planar subgraph. On the positive side, we show that any degree-2 contractible graph (as defined later) has a weak unit disk representation. In particular, any series-parallel graph is a weak unit disk representation.

We next study weak unit interval representations. It follows from [1] that all planar graphs with high girth are always weak unit interval graphs. We generalize the result by proving that graphs of bounded maximum average degree have weak unit interval representations for any given edge-labeling. In the other direction, we construct an example of a planar girth-4 graph which is not a weak unit interval, improving on the previously best girth-3 example. Furthermore, we show that dense planar graphs do not always admit such a weak unit interval graph representation.

Finally we study outerplanar graphs. It is known that some outerplanar graphs with girth 3 are not weak unit interval graphs, and our example of a girth-4 graph is not outerplanar. Thus, a natural question in this context is whether all girth-4 outerplanar graphs admit unit interval representation. We answer the question in a positive way: Every triangle-free outerplanar graph is a weak unit interval graph.

2 Weak Unit Disk Graph Representations

First we consider the complexity of recognizing weak unit disk graphs.

Lemma 1. *For a graph G with an edge-labeling ℓ , it is NP-hard to decide if G has a weak unit disk representation with respect to ℓ , even if the edges labeled N induce a planar subgraph.*

Proof. It is known that deciding if a planar graph is a unit disk graph is NP-hard [5]. Let n be the number of vertices of G , and define an edge-labeling ℓ of K_n by setting $\ell(e) = N$ if and only if e is an edge of G . Clearly, a unit disk representation of G is also a weak unit disk representation of K_n with respect to ℓ and vice versa. \square

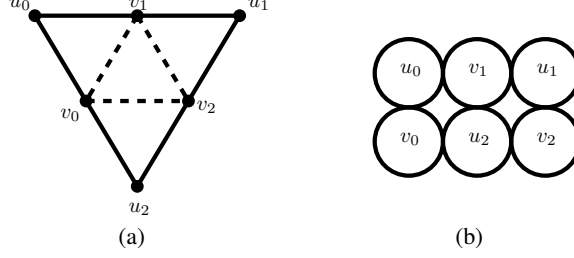


Fig. 2. (a) The sungraph has no weak unit interval representation, but (b) it has a weak unit disk representation. Near/far edges are indicated with solid/dashed line segments.

Note that the argument above only proves NP-hardness, and the problem of deciding if a graph with an edge-labeling has a weak unit disk representation is not known to be in NP. The obvious approach is to use a weak unit disk drawing as a polynomial size certificate. Unfortunately, it has recently been showed that unit disks graphs on n vertices may require $2^{2^{\Theta(n)}}$ bits for a unit disk drawing with integer coordinates [18].

Unit Disk Representation of Outerplanar and Related Graphs

Note that the class of weak unit disk graphs strictly contains the class of weak unit interval graphs. For example, in Fig. 2, we give a representation of the sungraph, which is not a weak unit interval graph. Our main goal here is to prove that every series-parallel graph is a weak unit disk graph, for every edge-labeling. To this end, we will study a larger class of graphs, which we call *degree-2 contractible* graphs. A simple graph G is a degree-2 contractible graph if one of the following holds:

1. G is an isolated vertex;
2. each component of G is a degree-2 contractible graph;
3. G has an edge (u, v) such that v has degree at most 2, and the graph obtained by contracting (u, v) and removing parallel edges is a degree-2 contractible graph.

Theorem 1. *Any degree-2 contractible graph is a weak unit disk graph.*

Proof. Suppose that G is a connected degree-2 contractible graph with $n > 1$ vertices and edge-labeling ℓ . Assume that G has no vertices of degree 1. We argue by induction that G is a weak unit disk graph. The base case is trivial, so consider the following inductive hypothesis. Suppose that every degree-2 contractible graph G' with $n - 1$ vertices has a weak unit disk representation I' with respect to any edge-labeling ℓ' of G' . Furthermore, suppose that (i) the disks of the representation have diameter 2, (ii) $I(v)$ has integer components for each vertex v of G' , and (iii) for each edge (u, v) of G' , $\|I(u) - I(v)\| \leq \sqrt{10}$.

Since G is degree-2 contractible, G has a vertex v with exactly two neighbors u and w , such that contracting the edge (u, v) results in a degree-2 contractible graph G' . We adopt the convention that, instead of contracting (u, v) , we delete v and add the edge (u, w) if it is not already present. By the inductive hypothesis, G' has a weak unit disk representation I' with respect to the edge-labeling ℓ restricted to the edges of G' (if edge (u, w) does not belong to G , give it an arbitrary label). Without loss of generality, we can assume that $I'(u) = (0, 0)$ and $I'(w) = (a, b)$ where $0 \leq b < a$. We construct a representation I of G by setting $I(x) = I'(x)$ for every vertex $x \neq v$. To compute the value of $I(v)$, consider Table 1. There we list for every possible value of $I(w)$, and every possible edge-labeling of (u, v) and (v, w) (except for one symmetric labeling), an appropriate value for $I(v)$ that satisfies the inductive hypothesis. The result follows. \square

Table 1. Details of the proof of Theorem 1. For every edge-labeling (up to symmetry) and every possible value of $I'(w)$, we give a value for $I'(v)$. In the column for the edge-labeling, an empty cell indicates to take the value from the cell above.

$\ell(u, v)$	$\ell(v, w)$	$I(w)$	$I(v)$	$\ell(u, v)$	$\ell(v, w)$	$I(w)$	$I(v)$	$\ell(u, v)$	$\ell(v, w)$	$I(w)$	$I(v)$
N	N	(1, 0)	(2, 0)	N	F	(1, 0)	(0, 2)	F	F	(1, 0)	(2, 2)
		(2, 0)	(1, 0)			(2, 0)	(0, 1)			(2, 0)	(1, 2)
		(3, 0)	(2, 0)			(3, 0)	(0, 1)			(3, 0)	(2, 2)
		(4, 0)	(2, 0)			(4, 0)	(1, 0)			(4, 0)	(2, 2)
		(1, 1)	(2, 0)			(1, 1)	(-1, 0)			(1, 1)	(0, 3)
		(2, 1)	(2, 0)			(2, 1)	(-1, 0)			(2, 1)	(0, 3)
		(3, 1)	(2, 0)			(3, 1)	(1, 0)			(3, 1)	(1, 2)
		(2, 2)	(2, 0)			(2, 2)	(1, 0)			(2, 2)	(0, 3)

Series-parallel graphs are defined as the graphs that do not have K_4 as a minor [7]. Hence by definition, these graphs are closed under edge contraction. It is also well-known that a series-parallel graph has a vertex of degree 2, and that every outerplanar graph is a subgraph of a series parallel graph. Thus, by Theorem 1, we have the following corollary.

Corollary 1. *Every outerplanar and series-parallel graph is a weak unit disk graph.*

3 Weak Unit Interval Graph Representations

In this section, we study weak unit interval representability, which is equivalent to threshold graph coloring [1]. Given a graph $G = (V, E)$, an edge-labeling $\ell : E \rightarrow \{N, F\}$, and integers $r > 0, t \geq 0$, G is said to be (r, t) -threshold-colorable with respect to ℓ if there exists a coloring $c : V \rightarrow \{1, \dots, r\}$ such that for each edge $(u, v) \in E$, $|c(u) - c(v)| \leq t$ if and only if $\ell(u, v) = N$. The coloring c is known as a *threshold-coloring*. It is easy to see that threshold-coloring is very similar to weak unit interval representation, with the only difference being that weak unit interval graphs do not require integer coordinates. We show that this does not matter.

Lemma 2. *A graph G is a weak unit interval graph with respect to an edge-labeling ℓ if and only if G is (r, t) -threshold-colorable with respect to ℓ for some integers $r > 0, t \geq 0$.*

Proof. Clearly, a threshold-coloring c is a weak unit interval representation of G with respect to ℓ (where we use t as the unit of the representation), so we need only show that a weak unit interval representation of G is equivalent to some threshold-coloring. Suppose that I is a weak unit interval representation of G with respect to ℓ . If any of the intervals of I intersect only at their endpoints, then we increase the length of each interval by some $\epsilon > 0$, and choose ϵ so that the intervals have rational length. Next, we perturb the center point of each interval, in some fixed order, by some ϵ so that each interval is centered at a rational point. Next, we scale the representation so that the center of each interval is an integer, and the length of the intervals is an integer. The modified representation is a threshold-coloring (although r and t may be large). \square

We now present a method for representing graphs, which admit a decomposition into a forest and a 2-independent set. By $G[U]$ we mean the subgraph of G induced by the vertex set $U \subseteq V$. Recall that a subset \mathcal{I} of vertices in a graph G is called *independent* if $G[\mathcal{I}]$ has no edges. \mathcal{I} is called *2-independent* if the shortest path in G between any 2 vertices of \mathcal{I} has length greater than 2. Similar decompositions have been applied to other graph coloring problems [2, 3, 19].

Lemma 3. Suppose $G = (\mathcal{I} \cup \mathcal{F}, E)$ is a graph such that \mathcal{I} is 2-independent, $G[\mathcal{F}]$ is a forest, and $\mathcal{I} \cap \mathcal{F} = \emptyset$. Then G is a weak unit interval graph.

Proof. We assume that all the intervals are centered at integer coordinates and have length $d = 1$. Suppose $\ell : E \rightarrow \{N, F\}$ is an edge-labeling. For each $v \in \mathcal{I}$, set $I(v) = 0$. Each vertex in $G[\mathcal{F}]$ is assigned a point from $\{-2, -1, 1, 2\}$ as follows. Choose a component T of $G[\mathcal{F}]$, and select a root vertex w of T . If w is far from a neighbor in \mathcal{I} , set $I(w) = 2$; otherwise, $I(w) = 1$. Now perform breadth first search on T , assigning an interval for each vertex as it is traversed. When we reach a vertex $u \neq w$, it has one neighbor t in T which has been processed, and at most one neighbor $v \in \mathcal{I}$. If v exists, we choose the interval $I(u) = 1$ if $\ell(u, v) = N$, and $I(u) = 2$ otherwise. Then, if the edge-label (u, t) is not satisfied, we multiply $I(u)$ by -1 . If v does not exist, we choose $I(u) = 1$ or -1 to satisfy the edge (u, t) . By repeating the procedure on each component of $G[\mathcal{F}]$, we construct a representation of G with respect to the labeling ℓ . \square

Recall that the *maximum average degree* of a graph G , denoted $\text{mad}(G)$, is the average vertex degree of a subgraph H with highest average degree. It is known that every planar graph G of maximum average degree $\text{mad}(G)$ strictly less than $\frac{26}{11}$ can be decomposed into a 2-independent set and a forest [6]. Hence,

Theorem 2. Every planar graph G with $\text{mad}(G) < \frac{26}{11}$ is a weak unit interval graph.

We also note that a planar graph with girth g satisfies $\text{mad}(G) < \frac{2g}{g-2}$. Therefore, a planar graph with girth at least 13 has always a weak unit interval representation.

Next we present a generalization of Lemma 3, suitable for graphs which have an independent set that is in some sense nearly 2-independent. The strategy is to delete certain edges so the independent set becomes 2-independent, obtain a unit interval representation using Lemma 3, and then modify it so that it is a representation of the original graph. Formally, let \mathcal{I} be an independent set in a graph G . Suppose that for every vertex $v \in \mathcal{I}$, there is at most one vertex $u \in \mathcal{I}$ such that the distance between v and u in G is 2. Also suppose that there is only one 2-path (a path with 2 edges) connecting v to u . Then we call \mathcal{I} *nearly 2-independent*. The pair $\{u, v\}$ is called an \mathcal{I} -pair, and the edges of the path (u, x, v) connecting u and v are called *bad edges*, which are associated with the bad pair $\{u, v\}$; see Fig. 3.

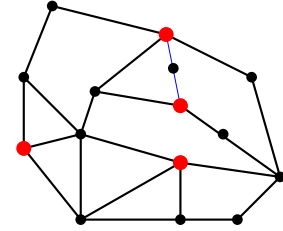


Fig. 3. Decomposition of a graph into a nearly 2-independent set (red vertices) and a forest (black vertices and edges). Thin blue edges are bad.

Lemma 4. Let $G = (\mathcal{I} \cup \mathcal{F}, E)$ be a graph such that \mathcal{I} is a nearly 2-independent set, $G[\mathcal{F}]$ is a forest and $\mathcal{I} \cap \mathcal{F} = \emptyset$. Then G is a weak unit interval graph with respect to ℓ .

Proof. Again, we assume that all the intervals are centered at integer coordinates. We use intervals of size $d = 3$.

Suppose that $\ell : E \rightarrow \{N, F\}$ is an edge-labeling of G . Let $E' \subseteq E$ be a set such that for each \mathcal{I} -pair $\{u, v\}$, exactly one of the bad edges associated with $\{u, v\}$ belongs to E' . Let $G' = (V, E - E')$. Then clearly \mathcal{I} is a 2-independent set in G' , and $G'[\mathcal{F}]$ is a forest, so there exists a weak unit disk representation I' of G' with respect to ℓ .

We now modify I' to construct a weak unit disk representation I of G with respect to ℓ . First, for each vertex $v \in V$, set $I(v) = 0$ if $I'(v) = 0$, $I(v) = 2$ if $I'(v) = 1$, and $I(v) = 5$ if $I'(v) = 2$ (if $I'(v)$ is negative, do the same but set $I(v)$ negative). It is clear that I is a weak unit disk representation of G' . Now, let $(x, y) \in E'$. One of these vertices, say x , is in \mathcal{I} so $I(x) = 0$, and $I(y) \in \{-5, -2, 2, 5\}$. Without loss of generality assume that $I(y) > 0$; the case where $I(y) < 0$ is symmetric. Now it is possible that $\ell(x, y) = N$

but $\|I(x) - I(y)\| > 3$ or that $\ell(x, y) = F$ but $\|I(x) - I(y)\| \leq 3$. In the first case, we must have $I(y) = 5$. We modify I so that $I(x) = 1$ and $I(y) = 4$. Note that y is still near to vertices with intervals centered at 2 or 5, and far from vertices with intervals centered at less than 1. Similarly, x is still close to the intervals at $-2, 0$, or 2 , but far from -5 and 5 . Thus all the edges of $E - E'$ are satisfied by the modification of I , and additionally the edge (x, y) is satisfied. In the second case, we have $I(y) = 2$. We modify I so that $I(x) = -1$ and $I(y) = 3$. As before, no edges which disagreed with the edge-labeling still disagree with the edge-labeling.

Since \mathcal{I} is nearly 2-independent, our modifications to the representation I will not affect non-local vertices, as every vertex in \mathcal{I} is adjacent to at most one edge of E' . \square

Weak Unit Interval Representation of Outerplanar Graphs

It is known that some outerplanar graphs containing triangles are not weak unit interval graphs, e.g., the sungraph in Fig. 2. Hence, we study weak unit interval representability of triangle-free outerplanar graphs. We start with a claim for graphs with girth 5.

Lemma 5. *An outerplanar graph with girth 5 is a weak unit interval graph.*

Proof. We prove that girth-5 outerplanar graphs may be decomposed into a forest and a 2-independent set using induction on the number of internal faces. The result will follow from Lemma 3. The claim is trivial for a single face, so assume that it is true for all girth-5 outerplanar graphs with $k \geq 1$ internal faces. Let G be a girth-5 outerplanar graph with $k + 1$ internal faces. Since G is outerplanar, it must have at least one face $f = (v_1, \dots, v_l)$, $l \geq 5$, such that every vertex of f except v_1, v_l is of degree 2. Consider the graph G' obtained by deleting v_2, \dots, v_{l-1} . G' has a decomposition into a 2-independent set \mathcal{I} and a forest T . Now we will add the vertices v_2, \dots, v_{l-1} to either \mathcal{I} or T so that \mathcal{I} is a 2-independent set in G , and T is a forest. If either of v_1, v_l belongs to \mathcal{I} , then add all the remaining vertices to T . Otherwise, add v_3 to \mathcal{I} and the rest to T . Since v_1, v_l are not in \mathcal{I} , v_3 has distance at least 3 from any other element of \mathcal{I} . \square

Next our goal is to show that a triangle-free outerplanar graph G always has a weak unit interval representation for any edge-labeling. We assume that all the intervals are centered at integer coordinates and we use intervals of size $d = 2$. Our strategy is to find a representation of G by a traversal of its weak dual graph G^* (the planar dual minus the outerface), where we find intervals for all the vertices in each interior face of G as it is traversed in G^* . Since we are considering triangle-free graphs, this implies that we take a path $P_n = (u_1, u_2, \dots, u_n)$, $n \geq 4$, where the two end vertices u_1 and u_n are already processed and we need to assign unit intervals to the internal vertices u_2, \dots, u_{n-1} of P_n . We additionally maintain the invariant in our representation that for each edge (u, v) of G , $\|I(u) - I(v)\| \leq 6$. For a particular edge-labeling ℓ of $P_n = (u_1, \dots, u_n)$, call a pair of coordinates x, y *feasible* if there is a weak unit disk representation I of P_n for ℓ with $d = 2$, where $I(u_1) = x$, $I(u_n) = y$, and for any $i \in \{1, \dots, n-1\}$, $\|I(u_i) - I(u_{i+1})\| \leq 6$. We first need the following three claims.

Claim 1 *For any value of $x \in \{2, 3, -2, -3\}$, the pair $0, x$ is feasible for any edge-labeling ℓ of $P_3 = (u_1, u_2, u_3)$.*

Proof. Without loss of generality, we may assume that $x > 0$. We compute a desired weak unit disk representation I with $r = 2$ for P_3 with respect to ℓ as follows. Assign $I(u_1) = 0$ and $I(u_3) = x$. Assign $I(u_2)$ in such a way that $|I(u_2)| = 2$ if $\ell(u_1, u_2) = N$, and $|I(u_2)| = 3$ if $\ell(u_1, u_2) = F$. Then choose the sign of $I(u_2)$ to be the same as $I(u_3)$ if $\ell(u_2, u_3) = N$, and the opposite of $I(u_3)$ if $\ell(u_2, u_3) = F$. \square

Claim 2 *For any edge-labeling of $P_3 = (u_1, u_2, u_3)$, either 0, 4 or 0, 6 are feasible.*

Proof. We compute a desired weak unit disk representation I with $r = 2$ for ℓ as follows. If $\ell(u_1, u_2) = \ell(u_2, u_3) = N$, then $I(u_1) = 0$, $I(u_2) = 2$, and $I(u_3) = 4$. Otherwise, assign $I(u_1) = 0$, $I(u_3) = 6$, and $I(u_2) = 2, 3$ or 4 in case the pair of edge-labelings $(\ell(u_1, u_2), \ell(u_2, u_3))$ have values (N, F) , (F, F) , and (F, N) , respectively. \square

Claim 3 *For any value of $x \in [-6, 6]$, the pair $0, x$ is feasible for any edge-labeling of $P_n = (u_1, u_2, \dots, u_n)$, $n \geq 4$.*

Proof. Without loss of generality, let $x \geq 0$. Consider first the case for $n = 4$. Take a particular edge-labeling ℓ of P_4 . For any value of $0 \leq x \leq 5$, there is at least one number $y \in \{2, 3, -2, -3\}$ and at least one number $z \in \{2, 3, -2, -3\}$ such that $|x - y| \leq 2$ and $2 < |x - z| \leq 6$. In particular, it suffices to choose for $x = 0$, $y = 2, z = 3$; for $x = 1, 2, 3, 4$, $y = 2, z = -2$ and for $x = 5$, $y = 3, z = 2$. Thus if $0 \leq I(u_4) \leq 5$, and regardless of whether $\ell(u_3, u_4)$ is N or F , one can choose a value for $I(u_3)$ from $\{2, 3, -2, -3\}$ respecting both the edge-labeling of (u_3, u_4) and the property that $\|I(u_3) - I(u_4)\| \leq 6$. Then by Claim 1, 0 and x is feasible for the edge-labeling ℓ of P_4 . A similar argument shows that if $\ell(u_3, u_4) = F$, then 0 and $x = 6$ is feasible. On the other hand, if $x = 6$ and $\ell(u_3, u_4) = N$, then both 4 and 6 are valid choices for $I(u_3)$. Then by Claim 2, 0 and 6 is feasible for any edge-labeling ℓ of P_4 .

Consider now the case with $n > 4$. Then assign coordinates $I(u_1) = 0$, $I(u_n) = x$ and for $i \in \{n - 1, \dots, 4\}$, assign $I(u_i) \in [-6, 6]$ such that it respects both $\ell(u_i, u_{i+1})$ and the property that $\|I(u_i) - I(u_{i+1})\| \leq 6$. Then a similar argument as that for $n = 4$ can be used to extend this representation to u_2 and u_3 . \square

The next corollary immediately follows from Claim 3.

Corollary 2. *Any pair x, y with $|x - y| \leq 6$, is feasible for any edge-labeling of $P_n = (u_1, u_2, \dots, u_n)$, $n \geq 4$.*

Theorem 3. *Every triangle-free outerplanar graph is a weak unit interval graph.*

Proof. If G is not 2-connected, we augment it in the following way. If G has a bridge (v, w) , let $u \neq w$ be a neighbor of v , and $x \neq v$ a neighbor of w , then add the edge (u, x) . If G has a cut vertex v , then let H_1, \dots, H_k be the 2-connected components of G containing v . For $i \in \{1, \dots, k - 1\}$, let u be a neighbor of v in H_i , and w be a neighbor of v in H_{i+1} . Add the path (u, x, w) , where x is a new vertex. Clearly, any weak unit interval representation of the new 2-connected graph is also a weak unit interval representation of G , and the new graph is outerplanar with girth 4.

Now let G be a 2-connected triangle-free outerplanar graph with $n > 4$ vertices embedded in the plane with every vertex on the outerface, and let ℓ be an edge-labeling of G . We next compute a weak unit interval representation of G for ℓ . The proof is by induction, with the n -vertex cycle as a base case. Assume the inductive hypothesis that every triangle-free outerplanar graph with fewer than n vertices is a weak unit interval graph. Further, assume that for such a graph G' and any edge-labeling ℓ' of G' , there is a weak unit interval representation of G' for ℓ' such that any 2 neighbor vertices u and v satisfy $\|I(u) - I(v)\| \leq 6$. It is clear that if G has at least two cycles, then G has a path $P_k = (u_1, \dots, u_k)$ where $k \geq 4$ such that $\deg(u_i) = 2$ for some $1 < i < k$. The theorem follows from the inductive hypothesis and Corollary 2. \square

Planar Graphs without Weak Unit Interval Representations

Planar graphs with high edge density may not have weak unit interval representations. First we prove the result for a *wheel graph*, defined as W_n , $n \geq 4$, formed by adding an edge from a vertex v_1 to every vertex of an $(n - 1)$ -cycle (v_2, \dots, v_n, v_2) .

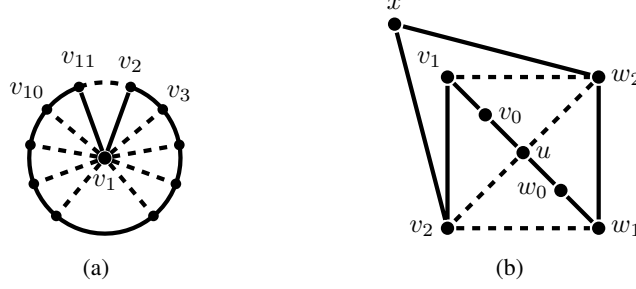


Fig. 4. (a) A wheel graph W_{11} with an edge-labeling, which does not have a weak unit interval representation. (b) A girth-4 graph with an edge-labeling, which does not have a weak unit interval representation.

Lemma 6. *A wheel graph is not a weak unit interval graph.*

Proof. Define an edge-labeling ℓ of W_n by $\ell(v_2, v_n) = F$, $\ell(v_1, v_i) = F$ for $3 \leq i \leq n-1$, and every other edge labeled N ; see Fig. 4(a). Suppose I is a weak unit interval representation of W_n with respect to ℓ . Since only one edge of the triangle (v_1, v_2, v_n, v_1) is far, it must be that $I(v_1) \neq I(v_2)$, so we may assume that $I(v_1) < I(v_2)$. For $3 \leq i \leq n$, if $I(v_{i-1}) > I(v_1)$, we have $I(v_i) > I(v_1)$, since $\ell(v_{i-1}, v_i) = N$ and either $\ell(v_1, v_{i-1})$ or $\ell(v_1, v_i)$ is F . But then $I(v_1) < I(v_2) \leq I(v_1) + 1$, and $I(v_1) < I(v_n) \leq I(v_1) + 1$, contradicting the fact that $\ell(v_2, v_n) = F$ and I is a weak unit interval representation. \square

Using Lemma 6, it is easy to see that any maximal planar graph with $|V| \geq 4$ is not a weak unit interval graph. Indeed, consider such a graph $G = (V, E)$ and a vertex $v \in V$; the neighborhood $N(v) = \{u \mid (v, u) \in E\}$ together with v induces a wheel subgraph. The observation leads to the following theorem.

Theorem 4. *Every planar graph G with $\text{mad}(G) \geq \frac{11}{2}$ is not a weak unit interval graph.*

Proof. To prove the claim, we show that a weak unit interval planar graph has at most $\lfloor 11|V|/4 \rfloor - 6$ edges.

Consider a vertex v of a weak unit interval planar graph $G = (V, E)$ and assume it is embedded in the plane. The neighborhood of v is acyclic; otherwise v and its neighborhood induce a wheel, which by Lemma 6 is not a weak unit interval graph. Thus the number of edges between any two neighbors of v is at most $\deg(v) - 1$, where $\deg(v)$ is the degree of v . Summing over all vertices, we get $S = 2|E| - |V|$. Let T and \bar{T} be the sets of triangular and non-triangular faces in an embedding of G . For each triangle $t \in T$ each of the edges in t is counted once in S . Thus, $2|E| - |V| \geq 3|T| \Rightarrow |T| \leq (2|E| - |V|)/3$. Counting both sides of the edges we get $2|E| \geq 3|T| + 4|\bar{T}| \Rightarrow |T| + |\bar{T}| \leq (2|E| + |T|)/4 \leq (8|E| - |V|)/12$, since $|T| \leq (2|E| - |V|)/3$. Thus, from Euler's formula $|V| - |E| + |T| + |\bar{T}| = 2$, we have $|V| - |E| + (8|E| - |V|)/12 \geq 2 \Rightarrow |E| \leq 11|V|/4 - 6$. \square

In [1] all examples of graphs without threshold-coloring (and thus, not weak unit interval graphs) have girth 3. We strengthen the bound by proving the following.

Lemma 7. *There exist planar girth-4 graphs that are not weak unit interval graphs.*

Proof. Consider the graph in Fig. 4(b). Suppose there exists a weak unit interval representation I . Without loss of generality suppose that $I(w_2) > I(u)$. Let us consider two cases. First, suppose $I(v_2) < I(u)$. Since the edges (u, v_2) and (u, w_2) are labeled F , it must be that $I(v_2) < I(u) - 1$ and $I(u) + 1 < I(w_2)$. Then vertex x must be represented by an interval near to both of these, which is impossible since $\|I(v_2) - I(w_2)\| > 2$.

Second, suppose $I(v_2) > I(u)$. Then $I(v_1) \geq I(v_2) - 1 > I(u)$, and $I(u) < I(w_2)$ implies that $I(v_1) < I(w_2)$. Similarly, $I(w_1) < I(v_2)$. Now, either $I(w_2) \leq I(v_2)$, or $I(v_2) < I(w_2)$. In the first case,

w_2 is near to v_1 since $I(v_1) < I(w_2) \leq I(v_2)$ and $\|I(v_1) - I(v_2)\| \leq 1$, a contradiction. The second case leads to a similar contradiction. \square

4 Conclusion and Open Problems

We studied weak unit disk and the weak unit interval representations for planar and outerplanar graphs. Many interesting open problems remain.

1. Deciding whether a graph is a weak unit disk (interval) graph with respect to a given edge-labeling is NP-complete. However, the problem of deciding whether a (planar) graph is a weak unit disk (interval) graph is open. Note that the class of weak unit disk (interval) graphs is not closed under minors, as subdividing each edge of a planar graph three times results in a planar graph with girth at least 10, which is a weak unit interval graph.
2. Tightening the lower and upper bounds for maximum average degree of weak unit interval graphs, given in Theorems 2 and 4, is a challenging open problem. Based on extensive computer experiments, we conjecture that there are no weak unit interval graphs with more than $2|V| - 3$ edges.
3. We considered planar graphs, but little is known for general graphs. In particular, it would be interesting to find out whether the edge density of weak unit disk (interval) graphs is always bounded by a constant.

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